# Derivative of Exponential Functions & Logarithmic Functions.

**Exponential function:** $f(x) = a^x$ , $a > 0$.

Find $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h}$$

Independent of $h$, so acts as a constant.

$$= \lim_{h\to 0} \frac{a^x (a^h - 1)}{h}$$

$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h} = \text{constant}$$

$$= a^x \cdot \log_e a = a^x \cdot (\ln a)$$

**Remark:** When $a = e$, then $[e^x]' = [e^x \cdot \ln e] = [e^x]$

**Find:** $[e^{7x}]'$, $[x^2 e^{-x}]'$, $[2^{\cos x}]'$, $[(e^x + e^{-x})^{-1}]'$

- $[e^{7x}]' = e^{7x} \cdot (7x)' = e^{7x} \cdot 7 = 7e^{7x}$
- $[x^2 e^{-x}]' = (x^2)' \cdot e^{-x} + x^2 \cdot (e^{-x})' = 2x e^{-x} + x^2(-e^{-x}) = (2x - x^2)e^{-x}$
- $[2^{\cos x}]' = (2^{\cos x} \cdot \ln 2) \cdot (\cos x)' = (2^{\cos x} \ln 2)(-\sin x) = -2^{\cos x} \cdot \ln 2 \cdot \sin x$
- $[(e^x + e^{-x})^{-1}]' = (-1)(e^x + e^{-x})^{-1-1} \cdot (e^x + e^{-x})' = -(e^x + e^{-x})^{-2} \cdot (e^x - e^{-x}) = -\dfrac{e^x - e^{-x}}{(e^x + e^{-x})^2}$

## Logarithmic Functions:

Since Exponential & Logarithmic functions are inverse to each other,

$$a^{\log_a x} = x$$

Let's take derivative on both side (wrt $x$):

$$\left[a^{\log_a x} \cdot \ln a\right]\left[\log_a x\right]' = 1$$

$$\Rightarrow \left[\log_a x\right]' = \frac{1}{\underbrace{a^{\log_a x}}_{x} \cdot \ln a} = \frac{1}{x \ln a}$$

So, when we replace $a$ by $e$, we get

$$\left[\ln x\right]' = \frac{1}{x}$$

Note: Domain of log functions are always $x > 0$ or $(0, \infty)$

Find. $\left[\log_5 (x^2)\right]''$, $\left[\ln\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right)\right]'$

- $\log_5 (x^2) = 2\log_5 x \Rightarrow \left[\log_5 (x^2)\right]'' = \left[2\log_5 x\right]''$

$$= 2\left[\log_5 x\right]''$$

$$= 2 \cdot \left[\frac{1}{x \ln 5}\right]'$$

$$= \frac{2}{\ln 5}\left[\frac{1}{x}\right]'$$

$$= \frac{2}{\ln 5}\left[-\frac{1}{x^2}\right]$$

$$\bullet \quad \ln\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right) = \ln\left(\sqrt[7]{x+1}\right) - \ln\left(x^7+1\right)$$

$$= \frac{1}{7}\ln(x+1) - \ln(x^7+1)$$

So $$\left[\ln\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right)\right]' = \left[\frac{1}{7}\ln(x+1) - \ln(x^7+1)\right]'$$

$$= \left[\frac{1}{7}\ln(x+1)\right]' - \left[\ln(x^7+1)\right]'$$

$$= \frac{1}{7}\left[\ln(x+1)\right]' - \frac{1}{x^7+1}\cdot\left[x^7+1\right]'$$

$$= \frac{1}{7(x+1)}\left[x+1\right]' - \frac{7x^6}{x^7+1}$$

$$= \frac{1}{7(x+1)} - \frac{7x^6}{x^7+1}$$

⊛ This problem can also be done by Chain Rule directly.

$$\left[\ln\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right)\right]' = \frac{1}{\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right)}\cdot\left[\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right)\right]'$$

$$= \frac{1}{\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right)}\cdot\frac{(x^7+1)\left[\sqrt[7]{x+1}\right]' - \left(\sqrt[7]{x+1}\right)\left[x^7+1\right]'}{(x^7+1)^2}$$

$$= \frac{1}{\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right)}\cdot\frac{(x^7+1)\left[(x+1)^{1/7}\right]' - \left(\sqrt[7]{x+1}\right)\left[7x^6\right]}{(x^7+1)^2}$$

$$= \frac{1}{\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right)}\cdot\frac{(x^7+1)\,\frac{1}{7}\cdot(x+1)^{1/7-1}\left[x+1\right]' - 7x^6\sqrt[7]{x+1}}{(x^7+1)^2}$$

$$= \frac{1}{\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right)} \cdot \frac{(x^7+1)\cdot\frac{1}{7}(x+1)^{-6/7}\cdot 1 - 7x^6\sqrt[7]{x+1}}{(x^7+1)^2}$$

$$= \frac{1}{\left(\frac{\sqrt[7]{x+1}}{x^7+1}\right)} \cdot \frac{\frac{1}{7}(x^7+1)^{-6/7} - 7x^6\sqrt[7]{x+1}}{(x^7+1)^2}$$